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A NOTE ON FIRST DEGCITY INDEX OF SOME SPECIAL GRAPHS

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Abstract. The first degcity index of a simple connected graph G is defined as $DC_1(G) = \sum_{uv \in E(G)} [e(u) + e(v)] [d(u) + d(v)]$, where $e(u)$ and $d(u)$ represents eccentricity and degree of a vertex u respectively. In this article, we compute the first degcity index for some special graphs viz. complete t-partite graph, friendship graph, broom graph, lollipop graph, double star graph, multi-star graph, Pl_n graph and square lattice graph.

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1. Introduction

Let $G = (V, E)$ be a simple connected graph, where $V = V(G)$ and $E = E(G)$ are called vertex set and edge set of G respectively. The degree of a vertex $v \in V(G)$ denoted by $d(v)$ or d_u is the number of edges incident with v. The distance $d(u, v)$ is the length of the shortest path between u and v in G and the eccentricity $e(u)$ or e_u of u is the maximum distance between u and all the other vertices of G. For standard terminologies and notions in graphs, one can refer [24].

Graph theory has established itself as a mathematical tool in a wide variety of subjects; one such subject is chemistry. Chemical graph theory is concerned with all aspects as the applications of graph theory to chemistry. A molecular graph is a connected graph whose vertices and edges represents the atoms and chemical bonds of a compound respectively. A graph invariant is a real number associated to a graph G which doesn't vary under the graph isomorphism. In chemistry, they are called as topological indices and they are derived from hydrogen suppressed molecular graphs.

Mathematicians and chemists have been conducting a long search to graph invariants which are related to degrees, distances etc., since they provide some information about physicochemical behaviors of chemical compounds. Numerous topological indices have been introduced and studied in the literature with some of them having wide applications in several areas of mathematics, chemistry and other areas of science. Today most of the research in graph theory is on introducing new topological indices and study them, refer

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[3, 7, 8, 12, 13, 14, 15, 16, 17, 19, 20]. Therefore, there is always a scope for defining new topological indices for graphs and to study them. Motivated by this, we have introduced new degree-eccentricity based topological indices called as degcity indices in [23] and studied the QSPR (Quantitative Structure-Property Relationship) of some physicochemical properties of polycyclic aromatic hydrocarbons (PAHs) using them. The QSPR analysis reveals that among all degcity indices, first, second, fourth, fifth and sixth indices correlates well with many properties. In this paper, we compute the first degcity index for some special graphs namely, complete t-partite graph, friendship graph, broom graph, lollipop graph, double star graph, multi-star graph, Pl_n graph and square lattice graph.
The first degcity index denoted by $DC_1(G)$ is defined as follows :

$$DC_1(G) = \sum_{uv \in E(G)} [e(u) + e(v)] [d(u) + d(v)] .$$

The first Zagreb index was introduced by I. Gutman and N. Trinajstić [5] and is defined as

$$M_1(G) = \sum_{u \in V(G)} d(u)^2.$$

In [4], T. Došlić et al. have showed that

$$M_1(G) = \sum_{u \in V(G)} d(u)^2 = \sum_{uv \in E(G)} [d(u) + d(v)]. \tag{1.1}$$

2. Main Results

We note the following elementary observations on the first degcity index.

Observation 2.1. *If K_n is a complete graph with $n \geq 2$ vertices, then*

$$DC_1(K_n) = 2n(n-1)^2.$$

Observation 2.2. *If P_n is a path with $n \geq 2$ vertices, then*

$$DC_1(P_n) = \begin{cases} 6n^2 - 16n + 12, & n \text{ is odd} \\ 6n^2 - 16n + 14, & n \text{ is even.} \end{cases}$$

Observation 2.3. *If C_n is a cycle with $n \geq 3$ vertices, then*

$$DC_1(C_n) = 8n \left\lfloor \frac{n}{2} \right\rfloor ,$$

where $\lfloor x \rfloor$ is the largest integer less than or equal to x.

Observation 2.4. *If $W_{1,n}$ is a wheel with $n+1$ vertices, then*

$$DC_1(W_{1,n}) = \begin{cases} 3n(n+11), & n > 3 \\ 72, & n = 3. \end{cases}$$

Observation 2.5. *If $K_{1,n}$ is a star with $n+1$ vertices $(n \geq 3)$, then*

$$DC_1(K_{1,n}) = 3n(n+1).$$

Observation 2.6. *If $K_{m,n}$ is complete bipartite graph, then*

$$DC_1(K_{m,n}) = 4mn(m+n).$$

Definition 2.1. [22] *A graph G is said to be a complete t-partite graph if there is a partition $V_1 \cup V_2 \cup \cdots \cup V_t = V(G)$ of the vertex-set, such that $uv \in E(G)$ if and only if u and v are in different parts of the partition. If $|V_i| = n_i$, $\forall\, 1 \leq i \leq t$, then G is denoted by $K_{n_1,n_2,\cdots,n_t}$.*

Theorem 2.1. *For a complete t-partite graph $G = K_{n_1,n_2,\ldots,n_t}$, where $t \geq 2$, $n_1 \geq n_2 \geq \ldots \geq n_t$ and $n = n_1 + n_2 + \ldots + n_t$*

$$DC_1(G) = 4\sum_{i=1}^{t} n_i(n - n_i)^2$$

Proof. Let $V_1, V_2, \cdots, V_t$ be the vertex sets of G such that $V(G) = V_1 \cup V_2 \cup \cdots \cup V_t$ with $|V_i| = n_i$ and $e(u) = 2, \forall\, u \in V_i$, $1 \leq i \leq t$. Also, if $u \in V_i$, then $d(u) = n_1 + n_2 + \cdots + n_{i-1} + n_{i+1} + \cdots + n_t = n - n_i$, $1 \leq i \leq t$. Then using (1.1), we have

$$\begin{aligned} DC_1(G) &= \sum_{uv \in E(G)} [e(u) + e(v)][d(u) + d(v)] \\ &= 4 \sum_{uv \in E(G)} [d(u) + d(v)] = 4 \sum_{u \in V(G)} [d(u)]^2 \\ &= 4\sum_{i=1}^{t} n_i(n - n_i)^2. \end{aligned}$$

□

Remark: If $n_1 = n_2 = \cdots = n_t = r$, then

$$DC_1(G) = 4\sum_{i=1}^{t} n_i(n - n_i)^2 = 4\sum_{i=1}^{t} r(n - r)^2 = 4rt(n - r)^2.$$

Definition 2.2. [10] *The friendship graph F_n is a graph obtained by joining n copies of the cycle C_3 with a common vertex.*

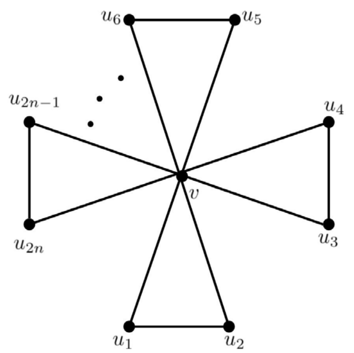

FIGURE 1. Friendship Graph

Theorem 2.2. *For a friendship graph $F_n, n \geq 2$,*

$$DC_1(F_n) = 4n(3n + 7).$$

Proof. A friendship graph $F_n, n \geq 2$ has (2n+1) vertices and 3n edges. Let $v, u_1, u_2, \cdots, u_{2n}$ be the vertices of F_n with v as the center vertex. Also, $d(v) = 2n, d(u_1) = d(u_2) = \cdots = d(u_{2n}) = 2$ and $e(v) = 1, e(u_1) = e(u_2) = \cdots = e(u_{2n}) = 2$. Now, we have

$$\begin{aligned}
DC_1(F_n) = &\sum_{uv \in E(F_n)} [e(u) + e(v)][d(u) + d(v)] \\
=&[e(v) + e(u_1)][d(v) + d(u_1)] + [e(v) + e(u_2)][d(v) + d(u_2)] + \cdots + \\
&+ [e(v) + e(u_{2n})][d(v) + d(u_{2n})] + [e(u_1) + e(u_2)][d(u_1) + d(u_2)] \\
&+ [e(u_3) + e(u_4)][d(u_3) + d(u_4)] + \cdots + [e(u_{2n-1}) + e(u_{2n})][d(u_{2n-1}) + d(u_{2n})] \\
=&2n(2 + 1)(2 + 2n) + n(2 + 2)(2 + 2) = 4n(3n + 7).
\end{aligned}$$

□

Definition 2.3. [9, 11] *The broom graph $B_{n,d}$ is a graph having a path P_d along with $(n - d)$ end vertices that are adjacent to the same end vertex of P_d.*

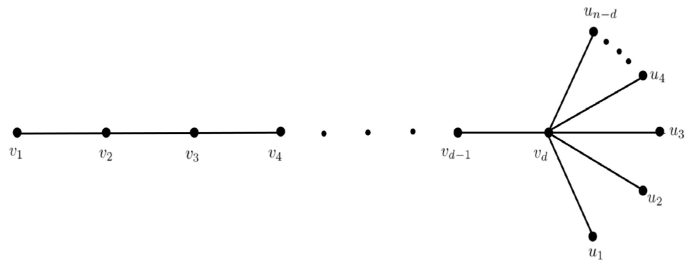

FIGURE 2. Broom Graph

Theorem 2.3. *For a Broom graph $B_{n,d}$*

$$DC_1(B_{n,d}) = \begin{cases} n^2(2d-1) - n(4d^2 - 8d + 5) + (2d^3 - d^2 - 7d + 10), & \text{if } d \text{ is odd} \\ n^2(2d-1) - n(4d^2 - 8d + 5) + (2d^3 - d^2 - 7d + 8), & \text{if } d \text{ is even} \end{cases}$$

Proof. Let $G = B_{n,d}$ and $\{v_1, v_2, \cdots, v_d, u_1, u_2, \cdots, u_{n-d}\}$ be the set of n vertices of the broom graph G. Then $d(v_1) = 1, d(v_2) = d(v_3) = \cdots = d(v_{d-1}) = 2, d(v_d) = n - d + 1$, $d(u_i) = 1$, $1 \leq i \leq d$ and $E(G) = n - 1$.

Case 1: If d is odd, then $e(v_1) = d, e(v_2) = d-1, e(v_3) = d-2, \cdots, e\left(v_{\frac{d+1}{2}-1}\right) = \dfrac{d+1}{2} + 1, e\left(v_{\frac{d+1}{2}}\right) = \dfrac{d+1}{2}, e\left(v_{\frac{d+1}{2}+1}\right) = \dfrac{d+1}{2}, e\left(v_{\frac{d+1}{2}+2}\right) = \dfrac{d+1}{2} + 1, \cdots, e(v_{d-1}) = d - 2, e(v_d) = d - 1$ and $e(u_i) = d$, $1 \leq i \leq n - d$. Now we have,

$$\begin{aligned}
DC_1(G) = &\sum_{uv \in E(G)} [e(u) + e(v)][d(u) + d(v)] \\
=&[e(v_1) + e(v_2)][d(v_1) + d(v_2)] + [e(v_2) + e(v_3)][d(v_2) + d(v_3)] \\
&+ \cdots + [e(v_{\frac{d+1}{2}-1}) + e(v_{\frac{d+1}{2}})][d(v_{\frac{d+1}{2}-1}) + d(v_{\frac{d+1}{2}})] \\
&+ [e(v_{\frac{d+1}{2}}) + e(v_{\frac{d+1}{2}+1})][d(v_{\frac{d+1}{2}}) + d(v_{\frac{d+1}{2}+1})]
\end{aligned}$$

$$\begin{aligned}
&+[e(v_{\frac{d+1}{2}+1})+e(v_{\frac{d+1}{2}+2})][d(v_{\frac{d+1}{2}+1})+d(v_{\frac{d+1}{2}+2})]\\
&+\cdots+[e(v_{d-2})+e(v_{d-1})][d(v_{d-2})+d(v_{d-1})]\\
&+[e(v_{d-1})+e(v_d)][d(v_{d-1})+d(v_d)]\\
&+\underbrace{[e(v_d)+e(u_1)][d(v_d)+d(u_1)]+\cdots+[e(v_d)+e(u_{n-d})][d(v_d)+d(u_{n-d})]}_{(n-d)-terms}\\
=&(d+d-1)(1+2)+(d-1+d-2)(2+2)+(d-2+d-3)(2+2)\\
&+\cdots+\left(\frac{d+1}{2}+1+\frac{d+1}{2}\right)(2+2)\\
&+\left(\frac{d+1}{2}+\frac{d+1}{2}\right)(2+2)\\
&+\left(\frac{d+1}{2}+\frac{d+1}{2}+1\right)(2+2)+\cdots\\
&+(d-3+d-2)(2+2)+(d-2+d-1)(2+n-d+1)\\
&+\underbrace{(d-1+d)(n-d+1+1)+\cdots+(d-1+d)(n-d+1+1)}_{(n-d)-terms}\\
=&(2d-1)3+8\underbrace{\left[(d-1+d-2)+(d-2+d-3)+\cdots+\left(\frac{d+1}{2}+1+\frac{d+1}{2}\right)\right]}_{\left(\frac{d-3}{2}\right)-terms}\\
&+4\left(\frac{d+1}{2}+\frac{d+1}{2}\right)+(2d-3)(n-d-1)+(n-d)(2d-1)(n-d+2)\\
=&n^2(2d-1)-n(4d^2-8d+5)+(2d^3-7d^2+13d+4)+8\sum_{k=1}^{\frac{d-3}{2}}(2d-2k-1)\\
=&n^2(2d-1)-n(4d^2-8d+5)+(2d^3-d^2-7d+10).
\end{aligned}$$

Case 2: If d is even, then $e(v_1)=d, e(v_2)=d-1, e(v_3)=d-2, \cdots, e\left(v_{\frac{d}{2}}\right)= e\left(v_{\frac{d}{2}+2}\right)=\frac{d}{2}+1, e\left(v_{\frac{d}{2}+1}\right)=\frac{d}{2}, \cdots, e(v_{d-1})=d-2, e(v_d)=d-1$ and $e(u_i)=d$, $1\le i\le n-d$. Now we have,

$$\begin{aligned}
DC_1(G)=&\sum_{uv\in E(G)}[e(u)+e(v)][d(u)+d(v)]\\
=&[e(v_1)+e(v_2)][d(v_1)+d(v_2)]+[e(v_2)+e(v_3)][d(v_2)+d(v_3)]\\
&+\cdots+[e(v_{\frac{d}{2}})+e(v_{\frac{d}{2}+1})][d(v_{\frac{d}{2}})+d(v_{\frac{d}{2}+1})]\\
&+[e(v_{\frac{d}{2}+1})+e(v_{\frac{d}{2}+2})][d(v_{\frac{d}{2}+1})+d(v_{\frac{d}{2}+2})]\\
&+\cdots+[e(v_{d-2})+e(v_{d-1})][d(v_{d-2})+d(v_{d-1})]\\
&+[e(v_{d-1})+e(v_d)][d(v_{d-1})+d(v_d)]\\
&+\underbrace{[e(v_d)+e(u_1)][d(v_d)+d(u_1)]+\cdots+[e(v_d)+e(u_{n-d})][d(v_d)+d(u_{n-d})]}_{(n-d)-terms}
\end{aligned}$$

$$
\begin{aligned}
=&(d+d-1)(1+2)+(d-1+d-2)(2+2)+(d-2+d-3)(2+2)\\
&+\cdots+\left(\frac{d}{2}+1+\frac{d}{2}\right)(2+2)+\left(\frac{d}{2}+\frac{d}{2}+1\right)(2+2)\\
&+\cdots+(d-3+d-2)(2+2)+(d-2+d-1)(2+n-d+1)\\
&+\underbrace{(d-1+d)(n-d+1+1)+\cdots+(d-1+d)(n-d+1+1)}_{(n-d)-terms}\\
=&(2d-1)3+8\underbrace{\left[(d-1+d-2)+(d-2+d-3)+\cdots+\left(\frac{d}{2}+1+\frac{d}{2}\right)\right]}_{\left(\frac{d}{2}-1\right)-terms}\\
&+(2d-3)(n-d-1)+(n-d)(2d-1)(n-d+2)\\
=&n^2(2d-1)-n(4d^2-8d+5)+(2d^3-7d^2+9d)+8\sum_{k=1}^{\frac{d}{2}-1}(2d-2k-1)\\
=&n^2(2d-1)-n(4d^2-8d+5)+(2d^3-d^2-7d+8)
\end{aligned}
$$

□

Definition 2.4. [9, 11] *The lollipop graph $L_{n,d}$ is a graph obtained from a path P_d and a complete graph K_{n-d} by joining one of the end vertices of P_d to all the vertices of K_{n-d}.*

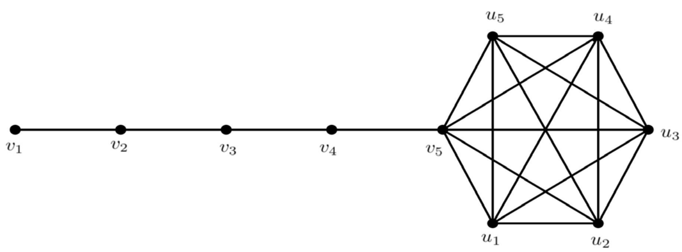

FIGURE 3. Lollipop Graph $L_{10,5}$

Theorem 2.4. *For a Lollipop graph $L_{n,d}$*

$$
DC_1(L_{n,d})=\begin{cases}
2dn^3-2(3d^2-d+1)n^2 & \\
+2(3d^3-2d^2+4d-2)n+(-2d^4+2d^3-8d+10), & \textit{if } d \textit{ is odd}\\
 & \\
2dn^3-2(3d^2-d+1)n^2 & \\
+2(3d^3-2d^2+4d-2)n+(-2d^4+2d^3-8d+8), & \textit{if } d \textit{ is even}
\end{cases}
$$

Proof. Let $G=L_{n,d}$ and $\{v_1,v_2,...,v_d,u_1,u_2,...,u_{n-d}\}$ be the set of n vertices of the lollipop graph G. Then $d(v_1)=1, d(v_2)=d(v_3)=...=d(v_{d-1})=2, d(v_d)=n-d+1$, $d(u_i)=n-d$, $1\le i\le d$ and $E(G)=(d-1)+\dfrac{(n-d+1)(n-d)}{2}=\dfrac{1}{2}\left[n^2-(2d-1)n+d-2\right]$.

Case 1: If d is odd, then $e(v_1)=d, e(v_2)=d-1, e(v_3)=d-2,\cdots, e\left(v_{\frac{d+1}{2}-1}\right)=\frac{d+1}{2}+1, e\left(v_{\frac{d+1}{2}}\right)=\frac{d+1}{2}, e\left(v_{\frac{d+1}{2}+1}\right)=\frac{d+1}{2}, e\left(v_{\frac{d+1}{2}+2}\right)=\frac{d+1}{2}+1,\cdots, e(v_{d-1})=d-2, e(v_d)=d-1$ and $e(u_i)=d$, $1\leq i\leq n-d$. Now we have,

$$
\begin{aligned}
DC_1(G)=&\sum_{uv\in E(G)}[e(u)+e(v)][d(u)+d(v)]\\
=&[e(v_1)+e(v_2)][d(v_1)+d(v_2)]+[e(v_2)+e(v_3)][d(v_2)+d(v_3)]\\
&+\cdots+[e(v_{d-1})+e(v_d)][d(v_{d-1})+d(v_d)]\\
&+[e(v_d)+e(u_1)][d(v_d)+d(u_1)]+\cdots+[e(v_d)+e(u_{n-d})][d(v_d)+d(u_{n-d})]\\
&+[e(u_1)+e(u_2)][d(u_1)+d(u_2)]+\cdots+[e(u_1)+e(u_{n-d})][d(u_1)+d(u_{n-d})]\\
&+[e(u_2)+e(u_3)][d(u_2)+d(u_3)]+\cdots+[e(u_2)+e(u_{n-d})][d(u_2)+d(u_{n-d})]\\
&+\cdots+[e(u_{n-d-1})+e(u_{n-d})][d(u_{n-d-1})+d(u_{n-d})]\\
=&(d+d-1)(1+2)+(d-1+d-2)(2+2)+(d-2+d-3)(2+2)+\cdots\\
&+\left(\frac{d+1}{2}+1+\frac{d+1}{2}\right)(2+2)+\left(\frac{d+1}{2}+\frac{d+1}{2}\right)(2+2)\\
&+\left(\frac{d+1}{2}+\frac{d+1}{2}+1\right)(2+2)+\cdots\\
&+(d-2+d-3)(2+2)+(d-2+d-1)(2+n-d+1)\\
&+(n-d)(d-1+d)(n-d+1+n-d)\\
&+\frac{(n-d)(n-d-1)}{2}(d+d)(n-d+n-d)\\
=&2dn^3-(6d^2-2d+2)n^2+(6d^3-4d^2+8d-4)n-2d^4+2d^3-6d^2+12d+4\\
&+8\sum_{k=1}^{\frac{d-3}{2}}(2d-2k-1)\\
=&2dn^3-2(3d^2-d+1)n^2+2(3d^3-2d^2+4d-2)n+(-2d^4+2d^3-8d+10).
\end{aligned}
$$

Case 2: If d is even, then $e(v_1)=d, e(v_2)=d-1, e(v_3)=d-2,\cdots, e\left(v_{\frac{d}{2}}\right)=\frac{d}{2}+1=e\left(v_{\frac{d}{2}+2}\right),\cdots, e\left(v_{\frac{d}{2}+1}\right)=\frac{d}{2},\cdots, e(v_{d-1})=d-2, e(v_d)=d-1$ and $e(u_i)=d, 1\leq i\leq n-d$. Now we have,

$$
\begin{aligned}
DC_1(G)=&\sum_{uv\in E(G)}[e(u)+e(v)][d(u)+d(v)]\\
=&[e(v_1)+e(v_2)][d(v_1)+d(v_2)]+[e(v_2)+e(v_3)][d(v_2)+d(v_3)]\\
&+\cdots+[e(v_{d-1})+e(v_d)][d(v_{d-1})+d(v_d)]\\
&+[e(v_d)+e(u_1)][d(v_d)+d(u_1)]+\cdots+[e(v_d)+e(u_{n-d})][d(v_d)+d(u_{n-d})]\\
&+[e(u_1)+e(u_2)][d(u_1)+d(u_2)]+\cdots+[e(u_1)+e(u_{n-d})][d(u_1)+d(u_{n-d})]\\
&+[e(u_2)+e(u_3)][d(u_2)+d(u_3)]+\cdots+[e(u_2)+e(u_{n-d})][d(u_2)+d(u_{n-d})]\\
&+\cdots+[e(u_{n-d-1})+e(u_{n-d})][d(u_{n-d-1})+d(u_{n-d})]\\
=&(d+d-1)(1+2)+(d-1+d-2)(2+2)+(d-2+d-3)(2+2)
\end{aligned}
$$

$$
\begin{aligned}
&+\cdots+\left(\frac{d}{2}+1+\frac{d}{2}\right)(2+2)+\left(\frac{d}{2}+\frac{d}{2}+1\right)(2+2)\\
&+\cdots+(d-3+d-2)(2+2)+(d-2+d-1)(2+n-d+1)\\
&+(n-d)(d-1+d)(n-d+1+n-d)\\
&+\frac{(n-d)(n-d-1)}{2}(d+d)(n-d+n-d)\\
=&2dn^3-(6d^2-2d+2)n^2+(6d^3-4d^2+8d-4)n-2d^4+2d^3-6d^2+8d\\
&+8\sum_{k=1}^{\frac{d}{2}-1}(2d-2k-1)\\
=&2dn^3-2(3d^2-d+1)n^2+2(3d^3-2d^2+4d-2)n+(-2d^4+2d^3-8d+8).
\end{aligned}
$$

□

Definition 2.5. [6, 11] *A double star graph $S_{n,m}$ is a graph obtained from $K_{1,n-1}$ and $K_{1,m-1}$ by joining their centers v_0 and u_0 such that the vertex set $V(S_{n,m}) = V(K_{1,n-1}) \cup V(K_{1,m-1}) = \{v_0, v_1, \cdots, v_{n-1}, u_0, u_1, \cdots, u_{m-1}\}$ and the edge set $E(S_{n,m}) = \{v_0u_0, v_0v_i, u_0u_j | 1 \le i \le (n-1); 1 \le j \le (m-1)\}$.*

Figure 4. Double Star Graph

Theorem 2.5. *For a double star graph $S_{n,m}$*

$$DC_1(S_{n,m}) = 5(n^2+m^2)+4(n+m)-10.$$

Proof. A double star graph $G = S_{n,m}$ has $(m+n)$ vertices and $(m+n-1)$ edges. Let $\{v_0, v_1, v_2, \cdots, v_{n-1}, u_0, u_1, u_2, \cdots, u_{m-1}\}$ be the vertices, then we have $d(v_0) = n, d(u_0) = m, d(v_i) = 1$, $1 \le i \le n-1, d(u_j) = 1$, $1 \le j \le m-1, e(v_0) = 2, e(u_0) = 2, e(v_i) = e(u_j) = 3$, $1 \le i \le n-1$, $1 \le j \le m-1$. Now we have,

$$
\begin{aligned}
DC_1(G) = &\sum_{uv\in E(G)}[e(u)+e(v)][d(u)+d(v)]\\
=&[e(v_0)+e(u_0)][d(v_0)+d(u_0)]\\
&+\underbrace{[e(v_0)+e(v_1)][d(v_0)+d(v_1)]+\cdots+[e(v_0)+e(v_{n-1})][d(v_0)+d(v_{n-1})]}_{(n-1)-terms}\\
&+\underbrace{[e(u_0)+e(u_1)][d(u_0)+d(u_1)]+\cdots+[e(u_0)+e(u_{n-1})][d(u_0)+d(u_{n-1})]}_{(m-1)-terms}
\end{aligned}
$$

$$=(2+2)(n+m)+\underbrace{(2+3)(n+1)+\cdots+(2+3)(n+1)}_{(n-1)-terms}$$
$$+\underbrace{(2+3)(m+1)+\cdots+(2+3)(m+1)}_{(m-1)-terms}$$
$$=5(n^2+m^2)+4(n+m)-10.$$

□

Definition 2.6. [18, 11] *Let $K_{1,n}$ be a star graph with vertex set $\{v_0, v_1, v_2, \cdots, v_n\}$. Introduce an edge to each of the pendant vertices $v_1, v_2, \cdots, v_n$ to get the desired graph $K_{1,n,n}$ with vertices $\{v_0, v_1, \cdots, v_n, v_{n+1}, \cdots, v_{2n}\}$, again introduce an edge to each of the pendant vertices $v_{n+1}, \cdots, v_{2n}$, to get the graph $K_{1,n,n,n}$. Continuing this $(m-1)$ times we get a graph $K_{1,\underbrace{n,n,\cdots,n}_{m-times}}$ known as multi-star graph with $(mn+1)$ vertices $v_0, v_1, v_2, \cdots, v_n, v_{n+1}, \cdots, v_{2n}, v_{2n+1}, \cdots, v_{3n}, \cdots, v_{(m-1)n+1}, \cdots, v_{mn}$ and mn edges.*

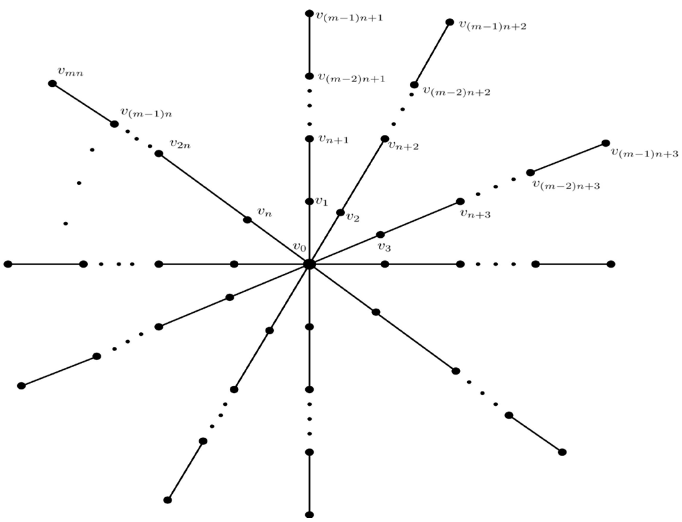

FIGURE 5. Multi-Star Graph

Theorem 2.6. *For a multi-star graph $G = K_{1,\underbrace{n,\cdots,n}_{m-times}}$*

$$DC_1(G) = n(12m^2+2mn+n-8m-1).$$

Proof. A multi-star graph $G = K_{1,\underbrace{n,\cdots,n}_{m-times}}$ has $(mn+1)$ vertices and mn edges. Let $v_0, v_1, \cdots, v_n, v_{n+1}, \cdots, v_{2n}, v_{2n+1}, \cdots, v_{3n}, \cdots, v_{(m-1)n}, v_{(m-1)n+1}, \cdots, v_{mn}$ be the vertices with v_0 as the center vertex, then we have $d(v_0) = n,\ d(v_p) = 2,\ 1 \le p \le (m-1)n,\ d(v_q) = 1,\ (m-1)n < q \le mn$,
$e(v_0) = m,\ e(v_i) = m+1,\ 1 \le i \le n,\ e(v_j) = m+2,\ n+1 \le j \le 2n,$

$e(v_k) = 2m$, $(m-1)n+1 \leq k \leq mn$. Now we have,

$$
\begin{aligned}
DC_1(G) = &\sum_{uv\in E(G)} [e(u)+e(v)][d(u)+d(v)] \\
=&[e(v_0)+e(v_1)][d(v_0)+d(v_1)] + [e(v_0)+e(v_2)][d(v_0)+d(v_2)] \\
&+\cdots+[e(v_0)+e(v_n)][d(v_0)+d(v_n)] \\
&+[e(v_1)+e(v_{n+1})][d(v_1)+d(v_{n+1})] + [e(v_2)+e(v_{n+2})][d(v_2)+d(v_{n+2})] \\
&+\cdots+[e(v_n)+e(v_{2n})][d(v_n)+d(v_{2n})] \\
&+[e(v_{n+1})+e(v_{2n+1})][d(v_{n+1})+d(v_{2n+1})] + [e(v_{n+2})+e(v_{2n+2})][d(v_{n+2})+d(v_{2n+2})] \\
&+\cdots+[e(v_{2n})+e(v_{3n})][d(v_{2n})+d(v_{3n})]+ \\
&\vdots \\
&+[e(v_{(m-2)n+1})+e(v_{(m-1)n+1})][d(v_{(m-2)n+1})+d(v_{(m-1)n+1})] \\
&+[e(v_{(m-2)n+2})+e(v_{(m-1)n+2})][d(v_{(m-2)n+2})+d(v_{(m-1)n+2})] \\
&+\cdots+[e(v_{(m-1)n})+e(v_{mn})][d(v_{(m-1)n})+d(v_{mn})] \\
=&\underbrace{(m+m+1)(n+2)+\cdots+(m+m+1)(n+2)}_{n-terms} \\
&+\underbrace{(m+1+m+2)(2+2)+\cdots+(m+1+m+2)(2+2)}_{n-terms} \\
&+\underbrace{(m+2+m+3)(2+2)+\cdots+(m+2+m+3)(2+2)}_{n-terms} \\
&\vdots \\
&+\underbrace{(m+m-1+m+m)(2+1)+\cdots++(m+m-1+m+m)(2+1)}_{n-terms} \\
=&n(2m+1)(n+2)+\underbrace{n(2m+3)4+\cdots+n(2m+2m-3)4}_{(m-2)-terms}+n(2m+2m-1)3 \\
=&n(2m+1)(n+2)+4n(m-2)3m+3n(4m-1) \\
=&n(12m^2+2mn+n-8m-1).
\end{aligned}
$$

□

Definition 2.7. [18, 11] *A $Pl_n (n \geq 3)$ graph is a graph constructed by joining a path P_{n-2} with P_2.*

Theorem 2.7. *For a Pl_n $(n \geq 6)$ graph*

$$DC_1(Pl_n) = 6(n^2+7n-26).$$

Proof. A Pl_n $(n \geq 6)$ graph has n vertices and $3(n-2)$ edges. Let $v_1, v_2, u_1 \cdots, u_{n-2}$ be the vertices of Pl_n. Then we have $d(v_1) = d(v_2) = n-1, d(u_1) = d(u_{n-2}) = 3, d(u_2) = \cdots = d(u_{n-3}) = 4, e(v_1) = e(v_2) = 1$ and $e(u_1) = e(u_2) = \cdots = e(u_{n-2}) = 2$. Now we have,

$$DC_1(Pl_n) = \sum_{uv\in E(Pl_n)} [e(u)+e(v)][d(u)+d(v)]$$

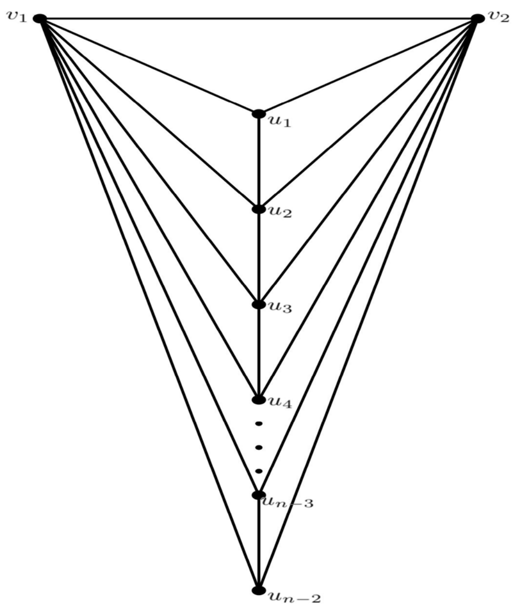

FIGURE 6. Pl_n Graph

$$\begin{aligned}
=&[e(v_1)+e(v_2)][d(v_1)+d(v_2)]\\
&+[e(v_1)+e(u_1)][d(v_1)+d(u_1)]+[e(v_1)+e(u_2)][d(v_1)+d(u_2)]+\cdots\\
&+[e(v_1)+e(u_{n-3})][d(v_1)+d(u_{n-3})]+[e(v_1)+e(u_{n-2})][d(v_1)+d(u_{n-2})]\\
&+[e(v_2)+e(u_1)][d(v_2)+d(u_1)]+[e(v_2)+e(u_2)][d(v_2)+d(u_2)]+\cdots\\
&+[e(v_2)+e(u_{n-3})][d(v_1)+d(u_{n-3})]+[e(v_2)+e(u_{n-2})][d(v_1)+d(u_{n-2})]\\
&+[e(u_1)+e(u_2)][d(u_1)+d(u_2)]+[e(u_2)+e(u_3)][d(u_2)+d(u_3)]+\cdots\\
&+[e(u_{n-4})+e(u_{n-3})][d(u_{n-4})+d(u_{n-3})]\\
&+[e(u_{n-3})+e(u_{n-2})][d(u_{n-3})+d(u_{n-2})]
\end{aligned}$$

$$\begin{aligned}
=&(1+1)(n-1+n-1)+(1+2)(n-1+3)\\
&+\underbrace{(1+2)(n-1+4)+\cdots+(1+2)(n-1+4)}_{(n-4)-terms}+(1+2)(n-1+3)\\
&+(1+2)(n-1+3)+\underbrace{(1+2)(n-1+4)+\cdots+(1+2)(n-1+4)}_{(n-4)-terms}\\
&+(1+2)(n-1+3)+(2+2)(3+4)\\
&+\underbrace{(2+2)(4+4)+\cdots+(2+2)(4+4)}_{(n-5)-terms}+(2+2)(4+3)\\
=&2(2n-2)+12(n+2)+6(n-4)(n+3)+32(n-5)+56\\
=&6(n^2+7n-26).
\end{aligned}$$

□

Definition 2.8. [21] *A square lattice graph SL_n is a $n \times n$ graph with n^2 vertices and $2n(n-1)$ edges.*

It has attracted the attention of many researchers because of its symmetric nature of its topology. For more detials on this, refer [1, 2].

Theorem 2.8. *For a square lattice graph SL_n*

$$DC_1(SL_n) = \begin{cases} 48n^3 - 130n^2 + 104n - 18, & \text{if } n \text{ is odd} \\ 48n^3 - 130n^2 + 120n - 32, & \text{if } n \text{ is even} \end{cases}$$

Proof. We know that the square lattice SL_n has n^2 vertices and $2n(n-1)$ edges. The edge partition of SL_n with respect to the degree of end vertices of each edge is given below [21]:

$(d(u), d(v))$, $uv \in E(SL_n)$	**Total number of edges**
(2,3)	8
(3,3)	$4(n-3)$
(3,4)	$4(n-2)$
(4,4)	$2n^2 - 10n + 12$

Case 1: Let n be odd and $G = SL_n$. Let

$$\begin{gathered} v_{11}, v_{12}, \cdots, v_{1(\frac{n+1}{2})}, \cdots, v_{1(n-1)}, v_{1n} \\ v_{21}, v_{22}, \cdots, v_{2(\frac{n+1}{2})}, \cdots, v_{2(n-1)}, v_{2n} \\ \vdots \\ v_{(\frac{n+1}{2})1}, v_{(\frac{n+1}{2})2}, \cdots, v_{(\frac{n+1}{2})(\frac{n+1}{2})}, \cdots, v_{(\frac{n+1}{2})(n-1)}, v_{(\frac{n+1}{2})n} \\ \vdots \\ v_{n1}, v_{n2}, \cdots, v_{n(\frac{n+1}{2})}, \cdots, v_{n(n-1)}, v_{nn} \end{gathered}$$

be the vertices of of G. Then

$$\begin{aligned} DC_1(G) = &\sum_{uv \in E(G)} [e(u) + e(v)][d(u) + d(v)] \\ =& DC_{2,3}(G) + DC_{3,3}(G) + DC_{3,4}(G) + DC_{4,4}(G), \end{aligned}$$

where

$$DC_{2,3} = \sum_{uv \in E_{2,3}(G)} [e(u) + e(v)][d(u) + d(v)],$$

$$DC_{3,3} = \sum_{uv \in E_{3,3}(G)} [e(u) + e(v)][d(u) + d(v)],$$

$$DC_{3,4} = \sum_{uv \in E_{3,4}(G)} [e(u) + e(v)][d(u) + d(v)],$$

$$DC_{4,4} = \sum_{uv \in E_{4,4}(G)} [e(u) + e(v)][d(u) + d(v)]$$

and

$$E_{2,3}(G) = \{uv \in E(G) | d(u) = 2 \ \ and \ \ d(v) = 3\},$$
$$E_{3,3}(G) = \{uv \in E(G) | d(u) = 3 \ \ and \ \ d(v) = 3\},$$
$$E_{3,4}(G) = \{uv \in E(G) | d(u) = 3 \ \ and \ \ d(v) = 4\},$$
$$E_{4,4}(G) = \{uv \in E(G) | d(u) = 4 \ \ and \ \ d(v) = 4\}.$$

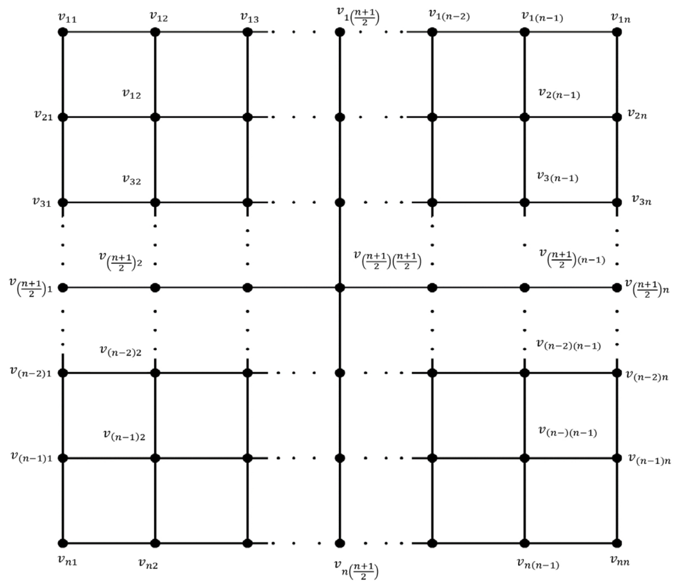

FIGURE 7. Square lattice graph with odd number of vertices

From Figure 7,we have

$$d(v_{11}) = d(v_{1n}) = d(v_{n1}) = d(v_{nn}) = 2,$$
$$d(v_{1k}) = d(v_{nk}) = d(v_{k1}) = d(v_{kn}) = 3, \ \forall\ 2 \le k < n,$$
$$d(v_{ij}) = 4, \ \forall\ 2 \le i, j < n$$

$$e(v_{11}) = 2n-2, e(v_{12}) = 2n-3, \cdots, e(v_{1(\frac{n-1}{2})}) = \frac{3n-1}{2}, e(v_{1(\frac{n+1}{2})}) = \frac{3n-3}{2},$$
$$e(v_{1(\frac{n+3}{2})}) = \frac{3n-1}{2}, \cdots, e(v_{1(n-1)}) = 2n-3, e(v_{1n}) = 2n-2,$$

$$e(v_{21}) = 2n-3, e(v_{22}) = 2n-4, \cdots, e(v_{2(\frac{n-1}{2})}) = \frac{3n-3}{2}, e(v_{2(\frac{n+1}{2})}) = \frac{3n-5}{2},$$
$$e(v_{2(\frac{n+3}{2})}) = \frac{3n-3}{2}, \cdots, e(v_{2(n-1)}) = 2n-4, e(v_{2n}) = 2n-3,$$
$$\vdots$$
$$e(v_{(\frac{n+1}{2})1}) = \frac{3n-3}{2}, e(v_{(\frac{n+1}{2})2}) = \frac{3n-5}{2}, \cdots, e(v_{(\frac{n+1}{2})(\frac{n-1}{2})}) = n, e(v_{(\frac{n+1}{2})(\frac{n+1}{2})}) = n-1,$$

$$e(v_{(\frac{n+1}{2})(\frac{n+3}{2})}) = n, \cdots, e(v_{(\frac{n+1}{2})(n-1)}) = \frac{3n-5}{2}, e(v_{(\frac{n+1}{2})n}) = \frac{3n-3}{2},$$
$$\vdots$$
$$e(v_{n1}) = 2n-2, e(v_{n2}) = 2n-3, \cdots, e(v_{n(\frac{n-1}{2})}) = \frac{3n-1}{2}, e(v_{n(\frac{n+1}{2})}) = \frac{3n-3}{2},$$
$$e(v_{n(\frac{n+3}{2})}) = \frac{3n-1}{2}, \cdots, e(v_{n(n-1)}) = 2n-3, e(v_{nn}) = 2n-2.$$

Now we have,

$$\begin{aligned}DC_{2,3}(G) = &\sum_{uv\in E_{2,3}(G)} [e(u)+e(v)][d(u)+d(v)]\\ =&4\left\{[e(v_{11})+e(v_{12})][d(v_{11})+d(v_{12})] + [e(v_{11})+e(v_{21})][d(v_{11})+d(v_{21})]\right\}\\ =&4[(2n-2+2n-3)(2+3)+(2n-2+2n-3)(2+3)] = 40(4n-5).\end{aligned}$$

$$\begin{aligned}DC_{3,3}(G) = &\sum_{uv\in E_{3,3}(G)} [e(u)+e(v)][d(u)+d(v)]\\ =&8[e(v_{12})+e(v_{13})][d(v_{12})+d(v_{13})] + 8[e(v_{13})+e(v_{14})][d(v_{13})+d(v_{14})]\\ &+\cdots+8[e(v_{1(\frac{n-1}{2})})+e(v_{1(\frac{n+1}{2})})][d(v_{1(\frac{n-1}{2})})+d(v_{1(\frac{n+1}{2})})]\\ =&8[(2n-3)+(2n-4)](3+3)+8[(2n-4)+(2n-5)](3+3)\\ &+\cdots+8\left[\frac{3n-1}{2}+\frac{3n-3}{2}\right](3+3)\\ =&48[(4n-7)+(4n-9)+\cdots+(3n-2)] = 48\left(\frac{n-3}{4}\right)(7n-9)\\ =&12(7n^2-30n+27).\end{aligned}$$

$$\begin{aligned}DC_{3,4}(G) = &\sum_{uv\in E_{3,4}(G)} [e(u)+e(v)][d(u)+d(v)]\\ =&8[e(v_{12})+e(v_{22})][d(v_{12})+d(v_{22})] + 8[e(v_{13})+e(v_{23})][d(v_{13})+d(v_{23})]\\ &+\cdots+8[e(v_{1(\frac{n-1}{2})})+e(v_{2(\frac{n-1}{2})})][d(v_{1(\frac{n-1}{2})})+d(v_{2(\frac{n-1}{2})})]\\ &+4[e(v_{1(\frac{n+1}{2})})+e(v_{1(\frac{n+1}{2})})][d(v_{1(\frac{n+1}{2})})+d(v_{1(\frac{n+1}{2})})]\\ =&8[(2n-3)+(2n-4)](3+4)+8[(2n-4)+(2n-5)](3+4)\\ &+\cdots+8\left[\frac{3n-1}{2}+\frac{3n-3}{2}\right](3+4)+4\left[\frac{3n-3}{2}+\frac{3n-5}{2}\right](3+4)\\ =&56[(4n-7)+(4n-9)+\cdots+(3n-2)]+28(3n-4)\\ =&56\left(\frac{n-3}{4}\right)(7n-9)+28(3n-4) = 14(7n^2-24n+19).\end{aligned}$$

$$\begin{aligned}DC_{4,4}(G) = &\sum_{uv\in E_{4,4}(G)} [e(u)+e(v)][d(u)+d(v)]\\ =&8\left\{1[e(v_{22})+e(v_{23})][d(v_{22})+d(v_{23})] + 2[e(v_{23})+e(v_{24})][d(v_{23})+d(v_{24})]\right.\\ &\left.+\cdots+\left(\frac{n-3}{2}\right)[e(v_{2(\frac{n-1}{2})})+e(v_{2(\frac{n+1}{2})})][d(v_{2(\frac{n-1}{2})})+d(v_{2(\frac{n+1}{2})})]\right\}\end{aligned}$$

$$
\begin{aligned}
&+4\Big\{1[e(v_{(\frac{n+1}{2})(\frac{n+1}{2})})+e(v_{(\frac{n+1}{2})(\frac{n-1}{2})})][d(v_{(\frac{n+1}{2})(\frac{n+1}{2})})+d(v_{(\frac{n+1}{2})(\frac{n-1}{2})})]\\
&+3[e(v_{(\frac{n+1}{2})(\frac{n-1}{2})})+e(v_{(\frac{n+1}{2})(\frac{n-2}{2})})][d(v_{(\frac{n+1}{2})(\frac{n-1}{2})})+d(v_{(\frac{n+1}{2})(\frac{n-2}{2})})]\\
&+5[e(v_{(\frac{n+1}{2})(\frac{n-2}{2})})+e(v_{(\frac{n+1}{2})(\frac{n-3}{2})})][d(v_{(\frac{n+1}{2})(\frac{n-2}{2})})+d(v_{(\frac{n+1}{2})(\frac{n-3}{2})})]\\
&+\cdots+(n-4)1[e(v_{(\frac{n+1}{2})3})+e(v_{(\frac{n+1}{2})2})][d(v_{(\frac{n+1}{2})3})+d(v_{(\frac{n+1}{2})2})]\Big\}\\
=&8\{[(2n-4)+(2n-5)](4+4)+2[(2n-5)+(2n-6)](4+4)\\
&+3[(2n-6)+(2n-7)](4+4)+\cdots+\left(\frac{n-3}{2}\right)\left[\frac{3n-3}{2}+\frac{3n-5}{2}\right](4+4)\Big\}\\
&+4\{[(n-1)+n](4+4)+3[n+(n+1)](4+4)+5[(n+1)+(n+2)](4+4)\\
&+\cdots+(n-4)\left[\frac{3n-7}{2}+\frac{3n-5}{2}\right]\Big\}\\
=&64[(4n-9)+2(4n-11)+3(4n-13)+\cdots+\left(\frac{n-3}{2}\right)(3n-4)]\\
&+32[(2n-1)+3(2n+1)+5(2n+3)+\cdots+(n-4)(3n-6)]\\
=&64\sum_{k=1}^{\frac{n-3}{2}}k[(4n-2k)-7]+32\sum_{k=1}^{\frac{n-3}{2}}[(2k-1)(2n+2k-3)]\\
=&\frac{8}{3}(n-3)(n-1)(10n-17)+\frac{16(n-3)}{3}(4n^2-18n+17)\\
=&8(6n^3-39n^2+80n-51).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
DC_1(G)=&40(4n-5)+12(7n^2-30n+27)\\
&+14(7n^2-24n+19)+8(6n^3-39n^2+80n-51)\\
=&48n^3-130n^2+104n-18.
\end{aligned}
$$

Case 2: Let n be even and $G=SL_n$. Let

$$
\begin{gathered}
v_{11},v_{12},\cdots,v_{1(\frac{n}{2})},v_{1(\frac{n}{2}+1)},\cdots,v_{1(n-1)},v_{1n}\\
v_{21},v_{22},\cdots,v_{2(\frac{n}{2})},v_{2(\frac{n}{2}+1)},\cdots,v_{2(n-1)},v_{2n}\\
\vdots\\
v_{(\frac{n}{2})1},v_{(\frac{n}{2})2},\cdots,v_{(\frac{n}{2})(\frac{n}{2})},v_{(\frac{n}{2})(\frac{n}{2}+1)}\cdots,v_{(\frac{n}{2})(n-1)},v_{(\frac{n}{2})n}\\
v_{(\frac{n}{2}+1)1},v_{(\frac{n}{2}+1)2},\cdots,v_{(\frac{n}{2}+1)(\frac{n}{2})},v_{(\frac{n}{2}+1)(\frac{n}{2}+1)}\cdots,v_{(\frac{n}{2}+1)(n-1)},v_{(\frac{n}{2}+1)n}\\
\vdots\\
v_{n1},v_{n2},\cdots,v_{n(\frac{n}{2})},v_{n(\frac{n}{2}+1)}\cdots,v_{n(n-1)},v_{nn}
\end{gathered}
$$

be the vertices of of G. Then

$$DC_1(G) = \sum_{uv \in E(G)} [e(u) + e(v)][d(u) + d(v)]$$
$$= DC_{2,3}(G) + DC_{3,3}(G) + DC_{3,4}(G) + DC_{4,4}(G),$$

where

$$DC_{2,3} = \sum_{uv \in E_{2,3}(G)} [e(u) + e(v)][d(u) + d(v)],$$
$$DC_{3,3} = \sum_{uv \in E_{3,3}(G)} [e(u) + e(v)][d(u) + d(v)],$$
$$DC_{3,4} = \sum_{uv \in E_{3,4}(G)} [e(u) + e(v)][d(u) + d(v)],$$
$$DC_{4,4} = \sum_{uv \in E_{4,4}(G)} [e(u) + e(v)][d(u) + d(v)]$$

and

$$E_{2,3}(G) = \{uv \in E(G) | d(u) = 2 \ \ and \ \ d(v) = 3\},$$
$$E_{3,3}(G) = \{uv \in E(G) | d(u) = 3 \ \ and \ \ d(v) = 3\},$$
$$E_{3,4}(G) = \{uv \in E(G) | d(u) = 3 \ \ and \ \ d(v) = 4\},$$
$$E_{4,4}(G) = \{uv \in E(G) | d(u) = 4 \ \ and \ \ d(v) = 4\}.$$

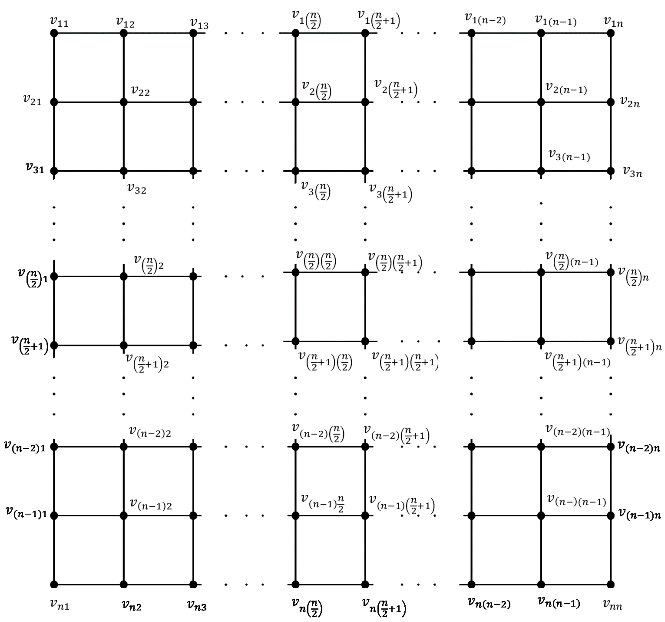

Figure 8. Square lattice with even number of vertices

From Figure 8, we have

$$d(v_{11}) = d(v_{1n}) = d(v_{n1}) = d(v_{nn}) = 2,$$
$$d(v_{1k}) = d(v_{nk}) = d(v_{k1}) = d(v_{kn}) = 3,\ \forall\ 2 \leq k < n,$$
$$d(v_{ij}) = 4,\ \forall\ 2 \leq i, j < n$$

$$e(v_{11}) = 2n{-}2, e(v_{12}) = 2n{-}3, \cdots, e(v_{1(\frac{n}{2}-1)}) = \frac{3n}{2}, e(v_{1(\frac{n}{2})}) = \frac{3n}{2}{-}1, e(v_{1(\frac{n}{2}+1)}) = \frac{3n}{2}{-}1,$$
$$e(v_{1(\frac{n}{2}+2)}) = \frac{3n}{2}, \cdots, e(v_{1(n-1)}) = 2n-3, e(v_{1n}) = 2n-2,$$

$$e(v_{21}) = 2n{-}3, e(v_{22}) = 2n{-}4, \cdots, e(v_{2(\frac{n}{2}-1)}) = \frac{3n}{2}{-}1, e(v_{2(\frac{n}{2})}) = \frac{3n}{2}{-}2, e(v_{2(\frac{n}{2}+1)}) = \frac{3n}{2}{-}2,$$
$$e(v_{2(\frac{n}{2}+2)}) = \frac{3n}{2} - 1, \cdots, e(v_{2(n-1)}) = 2n-4, e(v_{2n}) = 2n-3,$$
$$\vdots$$
$$e(v_{(\frac{n}{2})1}) = \frac{3n}{2}{-}1, e(v_{(\frac{n}{2})2}) = \frac{3n}{2}{-}2, \cdots, e(v_{(\frac{n}{2})(\frac{n}{2}-1)}) = n{+}1, e(v_{(\frac{n}{2})(\frac{n}{2})}) = n, e(v_{(\frac{n}{2})(\frac{n}{2}+1)}) = n,$$
$$e(v_{(\frac{n}{2})(\frac{n}{2}+2)}) = n+1, \cdots, e(v_{(\frac{n}{2})(n-1)}) = \frac{3n}{2} - 2, e(v_{(\frac{n}{2})n}) = \frac{3n}{2} - 1,$$

$$e(v_{(\frac{n}{2}+1)1}) = \frac{3n}{2} - 1, e(v_{(\frac{n}{2}+1)2}) = \frac{3n}{2} - 2, \cdots, e(v_{(\frac{n}{2}+1)(\frac{n}{2}-1)}) = n+1,$$
$$e(v_{(\frac{n}{2}+1)(\frac{n}{2})}) = n, e(v_{(\frac{n}{2}+1)(\frac{n}{2}+1)}) = n,$$
$$e(v_{(\frac{n}{2}+1)(\frac{n}{2}+2)}) = n+1, \cdots, e(v_{(\frac{n}{2}+1)(n-1)}) = \frac{3n}{2} - 2, e(v_{(\frac{n}{2}+1)n}) = \frac{3n}{2} - 1,$$
$$\vdots$$
$$e(v_{n1}) = 2n{-}2, e(v_{n2}) = 2n{-}3, \cdots, e(v_{n(\frac{n}{2}-1)}) = \frac{3n}{2}, e(v_{n(\frac{n}{2})}) = \frac{3n}{2}{-}1, e(v_{n(\frac{n}{2}+1)}) = \frac{3n}{2}{-}1,$$
$$e(v_{n(\frac{n}{2}+2)}) = \frac{3n}{2}, \cdots, e(v_{n(n-1)}) = 2n-3, e(v_{nn}) = 2n-2.$$

Now we have,

$$\begin{aligned}
DC_{2,3}(G) = &\sum_{uv \in E_{2,3}(G)} [e(u) + e(v)][d(u) + d(v)] \\
=&4\left\{[e(v_{11}) + e(v_{12})][d(v_{11}) + d(v_{12})] + [e(v_{11}) + e(v_{21})][d(v_{11}) + d(v_{21})]\right\} \\
=&4[(2n-2+2n-3)(2+3) + (2n-2+2n-3)(2+3)] = 40(4n-5).
\end{aligned}$$

$$\begin{aligned}
DC_{3,3}(G) = &\sum_{uv \in E_{3,3}(G)} [e(u) + e(v)][d(u) + d(v)] \\
&= 8[e(v_{12}) + e(v_{13})][d(v_{12}) + d(v_{13})] + 8[e(v_{13}) + e(v_{14})][d(v_{13}) + d(v_{14})] \\
&+ \cdots + 8[e(v_{1(\frac{n}{2}-1)}) + e(v_{1(\frac{n}{2})})][d(v_{1(\frac{n}{2}-1)}) + d(v_{1(\frac{n}{2})})] \\
&+ 4[e(v_{1(\frac{n}{2})}) + e(v_{1(\frac{n}{2}+1)})][d(v_{1(\frac{n}{2})}) + d(v_{1(\frac{n}{2}+1)})] \\
&= 8[(2n-3) + (2n-4)](3+3) + 8[(2n-4) + (2n-5)](3+3)
\end{aligned}$$

$$
\begin{aligned}
&+\cdots+8\left[\frac{3n}{2}+\frac{3n}{2}-1\right](3+3)+4\left[\frac{3n}{2}-1+\frac{3n}{2}-1\right](3+3)\\
&=48[\underbrace{(4n-7)+(4n-9)+\cdots+(3n-1)}_{\frac{n-4}{2} terms}]+24(3n-2)\\
&=48\left(\frac{n-4}{4}\right)(7n-8)+24(3n-2)=12(7n^2-30n+28).
\end{aligned}
$$

$$
\begin{aligned}
DC_{3,4}(G)=&\sum_{uv\in E_{3,4}(G)}[e(u)+e(v)][d(u)+d(v)]\\
=&8[e(v_{12})+e(v_{22})][d(v_{12})+d(v_{22})]+8[e(v_{13})+e(v_{23})][d(v_{13})+d(v_{23})]\\
&+\cdots+8[e(v_{1(\frac{n}{2})})+e(v_{2(\frac{n}{2})})][d(v_{1(\frac{n}{2})})+d(v_{2(\frac{n}{2})})]\\
=&8[(2n-3)+(2n-4)](3+4)+8[(2n-4)+(2n-5)](3+4)\\
&+\cdots+8\left[\frac{3n}{2}-1+\frac{3n}{2}-2\right](3+4)\\
=&56[\underbrace{(4n-7)+(4n-9)+\cdots+(3n-3)}_{\frac{n-2}{2} terms}]\\
=&56\left(\frac{n-2}{4}\right)(7n-10)=14(7n^2-24n+20).
\end{aligned}
$$

$$
\begin{aligned}
DC_{4,4}(G)=&\sum_{uv\in E_{4,4}(G)}[e(u)+e(v)][d(u)+d(v)]\\
=&8\left\{1[e(v_{22})+e(v_{23})][d(v_{22})+d(v_{23})]+2[e(v_{23})+e(v_{24})][d(v_{23})+d(v_{24})]\right.\\
&\left.+\cdots+\left(\frac{n-4}{2}\right)[e(v_{2(\frac{n}{2}-1)})+e(v_{2(\frac{n}{2})})][d(v_{2(\frac{n}{2}-1)})+d(v_{2(\frac{n}{2})})]\right\}\\
&+\left(\frac{n-4}{2}\right)[e(v_{2(\frac{n}{2})})+e(v_{3(\frac{n}{2})})][d(v_{2(\frac{n}{2})})+d(v_{3(\frac{n}{2})})]\\
&+\cdots+2[e(v_{(\frac{n}{2}-2)(\frac{n}{2})})+e(v_{(\frac{n}{2}-1)(\frac{n}{2})})][d(v_{(\frac{n}{2}-2)(\frac{n}{2})})+d(v_{(\frac{n}{2}-1)(\frac{n}{2})})]\\
&+1[e(v_{(\frac{n}{2}-1)(\frac{n}{2})})+e(v_{(\frac{n}{2})(\frac{n}{2})})][d(v_{(\frac{n}{2}-1)(\frac{n}{2})})+d(v_{(\frac{n}{2})(\frac{n}{2})})]\\
&+4\{[e(v_{2(\frac{n}{2})})+e(v_{2(\frac{n}{2}+1)})][d(v_{2(\frac{n}{2})})+d(v_{2(\frac{n}{2}+1)})]\\
&+[e(v_{3(\frac{n}{2})})+e(v_{3(\frac{n}{2}+1)})][d(v_{3(\frac{n}{2})})+d(v_{3(\frac{n}{2}+1)})]\\
&+\cdots+[e(v_{(\frac{n}{2})(\frac{n}{2})})+e(v_{(\frac{n}{2})(\frac{n}{2}+1)})][d(v_{(\frac{n}{2})(\frac{n}{2})})+d(v_{(\frac{n}{2})(\frac{n}{2}+1)})]\}\\
=&8\left\{[(2n-4)+(2n-5)](4+4)+2[(2n-5)+(2n-6)](4+4)\right.\\
&+3[(2n-6)+(2n-7)](4+4)+\cdots+\left(\frac{n-4}{2}\right)\left[\frac{3n}{2}-1+\frac{3n}{2}-2\right](4+4)\\
&+\left(\frac{n-4}{2}\right)\left[\frac{3n}{2}-2+\frac{3n}{2}-3\right](4+4)+\cdots+2[n+2+n+1](4+4)\\
&\left.+[n+1+n](4+4)\right\}\\
&+4\left\{\left[\frac{3n}{2}-2+\frac{3n}{2}-2\right](4+4)+\left[\frac{3n}{2}-3+\frac{3n}{2}-3\right]+\cdots+[n+n]\right\}
\end{aligned}
$$

$$
\begin{aligned}
=&64\{[1(4n-9)+2(4n-11)+3(4n-13)+\cdots+\left(\frac{n-4}{2}\right)(3n-3)] \\
&+\left(\frac{n-4}{2}\right)[(3n-5)+\cdots+2(2n+3)+1(2n+1)]\} \\
&+32[\underbrace{(3n-4)+(3n-6)+\cdots+2n}_{\frac{n-2}{2} terms}] \\
=&64\{\sum_{k=1}^{\frac{n-4}{2}} k[(4n-2k)-7]+\sum_{k=1}^{\frac{n-4}{2}} k[(2n+2k)-1]\}+32\left(\frac{n-2}{4}\right)(5n-4) \\
=&\frac{8}{3}(10n^3-75n^2+170n-120)+\frac{8}{3}(8n^3-57n^2+118n-72)+8(5n^2-14n+8) \\
=&8(6n^3-39n^2+82n-56).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
DC_1(G) =&40(4n-5)+12(7n^2-30n+28) \\
&+14(7n^2-24n+20)+8(6n^3-39n^2+82n-56) \\
=&48n^3-130n^2+120n-32.
\end{aligned}
$$

□

Conclusion

In this paper, the first degcity index for some special graphs viz. complete t-partite graph, friendship graph, broom graph, lollipop graph, double star graph, multi-star graph, Pl_n graph and square lattice graph has been computed analytically.

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